

ECE 532 - Lecture 3 — Linear independence, rank, $Ax=b$

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Linear independence is a property of a set of vectors.

If $v_1, v_2, \dots, v_n \in \mathbb{R}^m$, the set of linear combinations is:

$$S = \text{span}(v_1, \dots, v_n) = \left\{ \sum_{j=1}^n \alpha_j v_j \mid \alpha_j \in \mathbb{R} \right\}$$

The vectors $\{v_1, \dots, v_n\}$ are linearly independent if

$$\sum_{j=1}^n \alpha_j v_j = 0 \implies \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

Otherwise, they are linearly dependent.

if $n=m$, then
if $\{v_1, \dots, v_n\}$ are
linearly independent,
then $S = \mathbb{R}^n$

* if $\{v_1, \dots, v_n\}$ are linearly independent, they are called a basis for S .

Property 1: if $\{v_1, \dots, v_n\}$ is a basis, then every element of S can be represented as a unique linear combination of $\{v_1, \dots, v_n\}$.

Proof: by definition of S , each element $s \in S$ has at least one representation $s = \sum_{j=1}^n \alpha_j v_j$. Suppose there was some other representation, $s = \sum_{j=1}^n \beta_j v_j$.

$$\text{Then } \sum_{j=1}^n \alpha_j v_j = \sum_{j=1}^n \beta_j v_j \implies \sum_{j=1}^n (\alpha_j - \beta_j) v_j = 0.$$

Since the vectors $\{v_1, \dots, v_n\}$ are linearly independent, we conclude that $\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \dots = \alpha_n - \beta_n = 0$, so $\alpha_j = \beta_j$ for all j !

i.e. it's impossible to have two distinct representations. ■

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Property 2 : if $\{v_1, \dots, v_n\}$ are linearly dependent, then one of the v_i can be written as a linear combination of the other v_j 's.

Proof : since $\{v_1, \dots, v_n\}$ are not linearly independent, there exists some set of numbers $\{\alpha_1, \dots, \alpha_n\}$ such that they are not all zero, and with $\sum_{j=1}^n \alpha_j v_j = 0$. Let $\alpha_i \neq 0$ be one of the non-zero α_j 's. Write:

$$\sum_{j=1}^n \alpha_j v_j = \alpha_i v_i + \sum_{j \neq i} \alpha_j v_j = 0.$$

since $\alpha_i \neq 0$, we can divide by α_i and obtain:

$$v_i = \sum_{j \neq i} \left(-\frac{\alpha_j}{\alpha_i} \right) v_j$$

as required. ■

★ Example ($m=3, n=2$).

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ are linearly independent.}$$

★ Example ($m=3, n=3$)

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ are linearly independent.}$$

★ Example ($m=3, n=4$).

if we include $v_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, the vectors become linearly dependent.

$$\text{because: } v_4 = v_1 + v_2 - v_3.$$

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* Important: if a set of vectors is linearly dependent, it doesn't necessarily mean that every vector can be expressed as a linear combination of the others! Property 2 only says that one of them can be written this way!

example: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

we can write $v_2 = -v_3$ so they are linearly dependent.

However, v_1 cannot be written as a linear combination of v_2 and v_3 .

property 3: if $v_1, v_2, \dots, v_n \in \mathbb{R}^m$ then:

- (i) $\{v_1, \dots, v_n\}$ are linearly independent $\Rightarrow m \geq n$
- (ii) $m < n \Rightarrow \{v_1, \dots, v_n\}$ are linearly dependent.

Note: the reverse implications do not hold! i.e. if $\{v_1, \dots, v_n\}$ are linearly dependent, we can't conclude anything about m or n .

example: the zero matrix of any size has linearly dependent columns. e.g.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ both have dep. cols.}$$

Matrix rank

the rank of $A \in \mathbb{R}^{m \times n}$, denoted $\text{rank}(A)$, is the maximum number of linearly independent columns of A .

We already saw that if $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^m$ are linearly independent, then $m \geq k$. So $\text{rank}(A) \leq m$. Also, since A only has n columns, then $\text{rank}(A) \leq n$ obviously. Therefore:

$$\text{rank}(A) \leq \min(m, n).$$

★ if $\text{rank}(A) = \min(m, n)$, we say that A is "full rank"

Example: $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ has rank 2. (linearly indep cols).

$X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has rank 2. (col 3 is sum of col 1+2).

Example: if $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ then $xy^T \in \mathbb{R}^{m \times n}$ is rank 1.
not both zero.

Proof: $xy^T = x[y_1 \ y_2 \ \dots \ y_n] = \underbrace{[y_1x \ y_2x \ \dots \ y_nx]}_{\text{all multiples of the same column } x!}$

Convention:

$$\text{rank}(0) = 0$$

Matrix inverse

A square matrix $A \in \mathbb{R}^{n \times n}$ is said to be invertible if there exists some matrix $B \in \mathbb{R}^{n \times n}$ such that $AB = I$. We write $B = "A^{-1}"$ if it exists, we call B the inverse of A .

Properties

i) if $AB = I$ and A is invertible, then $BA = I$.

(can write $ABA = A \Rightarrow A(\underbrace{BA - I}) = 0$
must be zero (by combn. of cols of A))

ii) A is invertible if and only if A is full-rank.

(cols of A form a basis for \mathbb{R}^n).)

iii) A^{-1} is unique. (see if you can prove this).

another word for invertible is "non singular".

Similarly, a non-invertible matrix is called singular.

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Linear equations

$$Ax = b. \quad (\text{find the solution } x).$$

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 m × n n m.

* think of $Ax = [a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$
 (mixture of columns).

* what mixture of the columns of A gives me b ?

* is it true that $b \in \text{span}\{a_1, a_2, \dots, a_n\}$? ?
 (same as asking if a solution exists).

Simplest case : $m=n$. (A is square).

If A has linearly independent columns, then

- A is full rank
- columns of A form basis for $\text{span}\{a_1, \dots, a_n\}$, which is \mathbb{R}^n .
- so $b \in \text{span}\{a_1, \dots, a_n\}$, i.e. $Ax=b$ has a solution.
- there is exactly one solution to $Ax=b$.
- $x_{\text{soln}} = A^{-1}b$.

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Example :

$$A = \begin{bmatrix} \text{carbs(g)} & \text{fat(g)} & \text{protein(g)} \\ 25 & 0 & 1 \\ 20 & 1 & 2 \\ 40 & 1 & 6 \end{bmatrix}$$

frosted flakes
lucky charms
grape nuts

$$b = \begin{bmatrix} \text{calories} \\ 110 \\ 110 \\ 210 \end{bmatrix}$$

how many calories per gram
of carbs, fat, protein?
(should be 4, 9, 4).

Solve $Ax = b$.

$$\begin{pmatrix} 25 & 0 & 1 \\ 20 & 1 & 2 \\ 40 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 110 \\ 110 \\ 210 \end{pmatrix}$$

$$r_3 - 2r_2$$

$$\text{into } r_3$$

$$5r_2 - 4r_1$$

$$\text{into } r_2$$

$$5r_3 + r_2$$

$$\text{into } r_3$$

$$\begin{pmatrix} 25 & 0 & 1 \\ 20 & 1 & 2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 110 \\ 110 \\ -10 \end{pmatrix}$$

$$\begin{pmatrix} 25 & 0 & 1 \\ 0 & 5 & 6 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 110 \\ 110 \\ -10 \end{pmatrix}$$

$$\begin{pmatrix} 25 & 0 & 1 \\ 0 & 5 & 6 \\ 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 110 \\ 110 \\ 60 \end{pmatrix}$$

not 4, 9, 4 because
the cereal manufacturer
rounds the numbers!

Solve by back-substitution:

$$x_1 = 4.25, x_2 = 17.5, x_3 = 3.75.$$

General systems

$$A x = b$$

↑ ↑ ↑
 $\mathbb{R}^{m \times n}$ \mathbb{R}^n \mathbb{R}^m

- ★ if $m = n$, # equations = # unknowns.
 - ★ if $m > n$, More equations than unknowns.
 "Over determined" \rightarrow typically there is no solution.
 (can we find a solution that is "close"?).
 - ★ if $m < n$, Fewer equations than unknowns.
 "Under determined" \rightarrow infinitely many solutions.
 (among all solutions, which one should we pick?).
- $\{\text{Least squares}\}$
- $\{\text{Regularization}\}$.

We will look at these topics in more detail moving forward!